NOTE ON THE STABILITY PROBLEM FOR MAMMILLARY MATRICES

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ABSTRACT The importance of the stability problem for matrices of the special form called mammillary matrices has been noted by Hearon (1). The stability problem may be formulated in terms of the Liapounov matrix equation. A constructive procedure, dependent upon the particular structure of mammillary matrices, for the solution of the Liapounov equation and the subsequent solution of the stability problem is described in this note. For an $n \times n$ matrix the principal computational steps are the solution of an $n \times n$ linear system and the determination of the rank and signature of an $n \times n$ symmetric matrix.

INTRODUCTION

A real matrix M, which may be partitioned in the form

$$M = \begin{bmatrix} \alpha & r \\ c & \Delta \end{bmatrix}, \tag{1}$$

where α is a scalar, $r = (r_2 \cdots r_n)$ is a row vector, $c = (c_2 \cdots c_n)^T$ is a column vector and $\Delta = \text{diag}\ (\delta_2\delta_3 \cdots \delta_n)$, has been called a mammillary matrix (1). The importance of the stability problem for such matrices has been noted by Hearon (1).

It is known that the stability problem for any real matrix M may be completely solved in terms of the symmetric solution S of the Liapounov matrix equation

$$SM + M^TS = -I. \tag{2}$$

The situation is summarized by the following theorem (adapted from Ostrowski and Schneider, [2]):

Theorem 1. Suppose that the eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$ of the real $n \times n$ matrix M satisfy the condition

$$\prod_{i,j} (\lambda_i + \lambda_j) \neq 0 \quad \text{where} \quad 1 \leqq i, j \leqq n \tag{3}$$

and that P is a given positive definite matrix.

Then, the matrix equation

$$SM + M^T S = -P \tag{4}$$

admits an unique, symmetric solution S whose rank ρ and signature σ are related to the number $r(l)$ of eigenvalues of M lying in the right (left) half plane by the formulae

$$\rho + \sigma = 2r \qquad \rho - \sigma = 2l. \tag{5}$$

It is to be noted that, if the conditions of this theorem are not satisfied by M itself, a displacement of the origin along the real axis will produce a matrix $M + \mu I$ which does satisfy these conditions.

The solution of the Liapounov matrix equation 2, in the case in which M is in upper Hessenberg form, has been discussed elsewhere.[1] In this note it is shown that essentially the same solving procedure may be applied to mammillary matrices; and the required solution is obtained by a reasonably straightforward constructive procedure. A brief discussion of a method for the determination of the rank and signature of S concludes the paper.

PRELIMINARY RESULTS

A symmetric matrix S and a diagonal matrix D will be said to form a Liapounov pair with respect to M when they satisfy the matrix equation

$$SM + M^T S = 2D. \tag{6}$$

A symmetric matrix S is called a skewer (skew symmetrizer) of M when

$$SM = T + D \tag{7}$$

where T is skew-symmetric, $T^T = -T$, and D is diagonal. The following three lemmas may now be established; the proofs of the first two are detailed in the paper cited in footnote 1.

Lemma 1. Matrices S and D satisfy equation 6 when, and only when, they satisfy equation 7.

Lemma 2. If the condition of equation 3 is satisfied, and if $S_1, S_2 \cdots S_m$ and $D_1, D_2 \cdots D_m$ are Liapounov pairs, respectively, with respect to M, then when $S_1, S_2 \cdots S_m$ are linearly independent, so are $D_1, D_2 \cdots D_m$.

Lemma 3. If M is an $n \times n$ mammillary matrix which satisfies the condition of equation 3 and for which the components $c_2 \cdots c_n$ of c are all non-zero and the sums $\delta_i + \delta_j\,(i \neq j)$ of the diagonal elements of Δ are all non-zero, then there exist at least n linearly independent Liapounov pairs with respect to M.

[1] Howland, J. L., and J. A. Senez. A constructive method for the solution of the stability problem. Submitted for publication.

Proof. According to lemma 1 it is sufficient to characterize Liapounov pairs by the equation

$$S_j M = T_j + D_j . \tag{8}$$

It will be shown that, given the elements of the first row (column) of S_j, these equations uniquely determine the remaining elements of S_j. Upon selecting n linearly independent vectors for the first rows (columns) of the respective S_j, and applying lemma 2, n linearly independent Liapounov pairs may be obtained.

Upon partitioning S conformably with M, as described by equation 1, the matrix product

$$SM = \begin{bmatrix} \sigma & s_1^T \\ s_1 & \Sigma \end{bmatrix} \begin{bmatrix} \alpha & r \\ c & \Delta \end{bmatrix} = \begin{bmatrix} \sigma\alpha + s_1^T c & \sigma r + s_1^T \Delta \\ s_1 \alpha + \Sigma c & s_1 r + \Sigma\Delta \end{bmatrix}$$

is obtained, in which the symmetric $(n-1) \times (n-1)$ matrix Σ is the only unknown. Application of the requirement that this product have the form of equation 7 leads at once to the conclusion that the off-diagonal elements $\{s_{ij}\}$ of Σ are determined by the off-diagonal equations of the system

$$s_1 r + \Sigma\Delta = -(s_1 r + \Sigma\Delta)^T$$

while the diagonal elements $\{s_{ii}\}$ of Σ are determined by the system

$$s_1 \alpha + \Sigma c = -(\sigma r + s_1^T \Delta)^T.$$

For, from the first of these systems it is seen that

$$\Sigma\Delta + \Delta\Sigma = -(s_1 r + r^T s_1^T)$$

whence, in terms of components

$$s_{ij}\delta_j + \delta_i s_{ij} = -(s_{1i} r_j + r_i s_{1j}) \quad \text{where} \quad i \neq j$$

or

$$s_{ij} = -(s_{1i} r_j + r_i s_{1j})/(\delta_i + \delta_j) \quad \text{where} \quad i \neq j. \tag{9}$$

From the second of the above systems it is seen that

$$\Sigma c = -(\sigma r^T + \Delta s_1 + \alpha s_1)$$

or, in terms of components,

$$\sum_{k=2}^{n} s_{ik} c_k = -(\sigma r_i + \delta_i s_{1i} + \alpha s_{1i})$$

i.e.,

$$s_{ii} = -(\sum_{\substack{k=2 \\ k \neq i}}^{n} s_{ik}c_k + \sigma r_i + \delta_i s_{1i} + \alpha s_{1i})/c_i . \tag{10}$$

Under the hypotheses of the lemma, the equations 9 uniquely define the off-diagonal elements of Σ in terms of known quantities, and the equations 10 uniquely define the diagonal elements s_{ii}. The required Liapounov pair is, therefore,

$$S = \begin{bmatrix} \sigma & s_1^T \\ s_1 & \Sigma \end{bmatrix}, \qquad D = \operatorname{diag}\,(d_1 d_2 \cdots d_n) \tag{11}$$

where

$$d_1 = \sigma\alpha + s_1^T c \qquad d_j = s_{1j}r_j + s_{jj}\delta_j \quad (j > 1). \tag{12}$$

Upon repeating this calculation for n linearly independent choices of the vector $(\sigma, s_1)^T$, the required set of Liapounov pairs is obtained.

The proof of this lemma may be illustrated by the case $n = 4$, in which the equations 9 are

$$\begin{aligned} b &= -(r_2 y + r_3 x)/(\delta_2 + \delta_3) \\ c &= -(r_2 z + r_4 x)/(\delta_2 + \delta_4) \\ e &= -(r_3 z + r_4 y)/(\delta_3 + \delta_4) \end{aligned}$$

while the equations 10 are

$$\begin{aligned} a &= -(r_2 w + \delta_2 x + \alpha x + c_3 b + c_4 c)/c_2 \\ d &= -(r_3 w + \delta_3 y + \alpha y + c_2 b + c_4 e)/c_3 \\ j &= -(r_4 w + \delta_4 z + \alpha z + c_2 c + c_3 e)/c_4. \end{aligned}$$

Upon setting $(w\ x\ y\ z)^T$ equal in turn to, for example, the columns of the unit matrix, four linearly independent Liapounov pairs may be obtained.

COMPUTATIONAL PROCEDURE

The solution of equation 2 may now be sought as a linear combination

$$S = \sum_{j=1}^{n} p_j S_j \tag{13}$$

whence the coefficients p_j are defined by the nonsingular $n \times n$ linear system

$$\sum_{j=1}^{n} p_j D_j = -I/2. \tag{14}$$

For, since $S_j M + M^T S_j = 2D_j$ for $1 \leqq j \leqq n$, then

$$(\sum_{j=1}^{n} p_j S_j) M + M^T (\sum_{j=1}^{n} p_j S_j) = 2 \sum_{j=1}^{n} p_j D_j$$

so that, upon choosing the p_j to satisfy equation 14, the S defined by equation 13 will satisfy equation 2. The rank and signature of S then furnish the solution of the stability problem, according to the equations 5. The proposed computational procedure may be summarized as follows.

1. Select the n columns of the unit matrix as starting vectors. For each starting vector form the off-diagonal elements of Σ_j from equations 9 and the diagonal elements from equations 10. Assemble each S_j according to equation 11.

2. Compute the diagonal elements $(d_j^1 \cdots d_j^n)$ of each D_j according to equation 12 and construct the coefficient matrix $C = (d_j^i)$ of the linear system of equation 14 by assembling the diagonal elements of $D_1, D_2 \cdots D_n$ as respective columns of C. Solve the $n \times n$ linear system

$$Cp = -I/2 \tag{15}$$

equivalent to equation 14 for $p = (p_1, p_2 \cdots p_n)$.

3. Form S according to equation 13.

4. Determine the rank ρ and signature σ of S, and calculate r and l from equation 5.

DISCUSSION

The conditions $\delta_i + \delta_j \neq 0$ of lemma 3 may be guaranteed by a displacement of the origin along the real axis. It seems that, in some applications at least (see reference 1), the conditions $c_j \neq 0$ are automatically satisfied. If some $c_j = 0$ but all $r_j \neq 0$, the method may be applied to M^T in place of M. Otherwise this method could break down.

The coefficient matrix C of the linear system of equation 15, when it may be constructed at all, is known to be nonsingular. The condition of C with respect to this problem is, however, not known. It would seem that it will depend upon any real translations of the origin that may have been made. Moreover, applying the generalized Liapounov equation 4, it is seen that the $-I$ appearing in equations 14, 15 and elsewhere may be replaced with $-D$, where D is any diagonal matrix with positive diagonal elements. There is thus an opportunity to improve the condition of C by row-equilibration (see reference 3 for a discussion of this procedure), i.e. by multiplying individual rows by positive constants, replacing equation 15 with

$$DCp = -I. \tag{16}$$

It is convenient to transform S, with the aid of Householder transformations (see reference 4), to tridiagonal form in order to compute its rank and signature. The required computations are described in the following theorem, which is adapted from Givens (5).

Theorem 2. Suppose that S is a real, symmetric, $n \times n$, tridiagonal matrix with $s_{ii} = a_i$, $s_{ii+1} = s_{i+1i} = b_i$, $s_{ij} = 0$ when $|i - j| > 1$.

Let $d_0 = 1, b_0 = 0, \rho_0 = 0, \sigma_0 = 0$ and

$$\operatorname{sgn} x = \begin{cases} +1 & \text{when} \quad x > 0 \\ 0 & \text{when} \quad x = 0 \\ -1 & \text{when} \quad x < 0. \end{cases}$$

Let the sequences $\{d_i\}$, $\{\sigma_i\}$, $\{\rho_i\}$ be defined by the following recursions:

1. $d_i = a_i d_{i-1} - b_{i-1}^2 d_{i-2}$; $\sigma_i = \sigma_{i-1} + \operatorname{sgn}(d_i d_{i-1})$; $\rho_i = \rho_{i-1} + \operatorname{sgn} d_i^2$ when $d_{i-1} \neq 0$
2. $d_i = a_i$; $\sigma_i = \sigma_{i-1} + \operatorname{sgn} d_i$; $\rho_i = \rho_{i-1} + \operatorname{sgn} d_i^2$ when $d_{i-1} = 0$ and $b_{i-1} = 0$
3. $d_i = -b_{i-1}^2$; $\sigma_i = \sigma_{i-1}$; $\rho_i = \rho_{i-1} + 2$ when $d_{i-1} = 0, b_{i-1} \neq 0$ and $d_{i-2} = 0$.
4. $d_i = -b_{i-1}^2 d_{i-2}$; $\sigma_i = \sigma_{i-1}$; $\rho_i = \rho_{i-1} + 2$ when $d_{i-1} = 0, b_{i-1} \neq 0$ and $d_{i-2} \neq 0$.

Then, $\rho = \rho_n$ is the rank of S and $\sigma = \sigma_n$ is the signature of S.

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